Applied Numerical Mathematics 3 (1987) 361–362
North-Holland

CORRIGENDUM

J.G. BLOM, J.H.M. TEN THIJE BOONKKAMP and J.G. VERWER
Centre for Mathematics and Computer Science, Amsterdam, The Netherlands

"On the odd-even hopscotch scheme for the numerical integration of time-dependent partial differential equations", J.H.M. ten Thije Boonkkamp and J.G. Verwer [*Applied Numerical Mathematics 3* (1, 2) (1987) 183–193]

In Section 4 global Richardson extrapolation is suggested as a means for eliminating the Du Fort–Frankel deficiency. Then, in Section 5, on the basis of two numerical examples the conclusion is drawn that the deficiency is truly absent in the extrapolated scheme. This corrigendum serves to show that this conclusion has turned out to be incorrect.

First consider the linear heat flow equation $u_t = u_{xx}$ which is a special case of (4.1). Suppose that we compute a solution on successively finer grids, using the stepsizes $\tau = \tau_0, \frac{1}{2}\tau_0, \frac{1}{4}\tau_0, \ldots$ and $h = h_0, \frac{1}{2}h_0, \frac{1}{4}h_0, \ldots$, with either the hopscotch scheme or the Du Fort–Frankel scheme. It is thus assumed, through the equivalence property, that the two schemes generate the same approximate values. As $\tau, h \to 0$, these approximations converge to the solution values of the related problem $v_t = v_{xx} - (\tau^2/h^2)v_{tt}$. Let (x, t) be a point shared by all grids. Obviously $v(x, t) = v_{\tau/h}(x, t)$, that is, apart from initial and boundary data, $v(x, t)$ is determined exclusively bh the ratio of τ and h.

Let us now consider the suggested extrapolation procedure. In the limit, that is $\tau, h \to 0$ as above, we herewith form the values

$$w(x, t) = w_{\tau/h}(x, t) = \tfrac{4}{3}v_{\tau/h}(x, t) - \tfrac{1}{3}v_{2\tau/h}(x, t).$$

A trivial calculation shows that $w(x, t)$ is a solution of the differential equation

$$w_t = w_{xx} + \tfrac{4}{3}(\tau^2/h^2)(v_{2\tau/h} - v_{\tau/h})_{tt},$$

which still contains a (τ^2/h^2)-term. This implies that the extrapolation cannot have the effect which was aimed at. Also observe that $w_{\tau/h}(x, t) = w_{(\tau/2)/(h/2)}(x, t)$.

Next we consider the numerical example Problem 1 of Section 5 with the aim of presenting the correct interpretation of Table 2 and illustrating more comprehensively the effect of the extrapolation. For this purpose we show, in addition to Tables 1 and 2, the new Tables 1′, 2′ and 5. Their entries have the following meaning. Table 5 gives the accuracy obtained in the spatial discretization (5.1). Its entries contain the minimum of the number of significant digits in the space errors, i.e., $\min_j(-{}_{10}\log|u(x_j, 1) - U_j(1)|)$. Tables 1′ and 2′ give the accuracy obtained in the time integration of (5.1) by means of scheme (5.2) and the extrapolation thereof, respectively. Their entries contain the minimum of the number of significant digits in the time errors, i.e., $\min_j(-{}_{10}\log|U_j(1) - U_j^N|)$ for Table 1′ and $\min_j(-{}_{10}\log|U_j(1) - \tilde{U}_j^N|)$ for Table 2′. Recall that the old Tables 1, 2 refer to the full error, being the sum of the space error and the time error.

0168-9274/87/$3.50

Table 1′

τ^{-1}	h^{-1}				
	20	40	80	160	320
20	2.03				
40	2.63	2.03			
80	3.23	2.62	2.03		
160	3.84	3.22	2.62	2.03	
320	4.44	3.83	3.22	2.62	2.03
640	5.04	4.43	3.83	3.22	2.62
1280	5.64	5.03	4.43	3.83	3.22
2560	6.24	5.63	5.03	4.43	3.83

Table 2′

τ^{-1}	h^{-1}				
	20	40	80	160	320
40	2.93				
80	4.28	2.93			
160	5.53	4.28	2.93		
320	6.74	5.53	4.28	2.93	
640	7.95	6.74	5.52	4.28	2.93
1280	9.15	7.95	6.74	5.52	4.28
2560	10.37	9.15	7.95	6.74	5.52

Table 5

h^{-1}	20	40	80	160	320
	3.26	3.86	4.47	5.07	5.67

Inspection of Table 1′ reveals two relevant features of the hopscotch scheme (5.2), namely its second order in time for fixed h (N.B. ${}_{10}\log 2 \approx 0.3$) and the h^{-2}-dependence of the error function $e^{(2)}$ occurring in the global error expansion (4.4). Of course, this h^{-2}-dependence is due to the Du Fort–Frankel deficiency. In passing we note that when the numbers in Table 1 are slightly larger than the minima of the corresponding numbers in Tables 1′ and 5, that is due to cancellation of time and space errors. We next consider the more interesting Table 2′ of the extrapolated scheme. Surprisingly, this table shows fourth order in time for fixed h (the entries increase approximately with 1.2 upon halving of τ), which means that the error function $e^{(3)}$ in (4.4) is absent. However, it also shows a h^{-4}-dependence of the next error function $e^{(4)}$, which in turn implies that the deficiency is still there. This observation illustrates our proof above for the heat equation.

Comparison of Tables 2′ and 1′ clearly shows that the extrapolation does reduce the time integration errors. In fact, the decrease is so large that for many of the entries the spatial error becomes dominant. This explains why in the greater part of the full error Table 2 second order shows up upon simultaneously halving τ and h and, consequently, why the extrapolation in connection to the Du Fort–Frankel deficiency was misinterpreted.